A GENERALIZED DEFINITION OF LIMIT.*

By E. D. Roe, Jr.

The object of this paper is to suggest a logical point of view from which a generalization of the definition of limit may be secured.

After some discussion it is suggested to so phrase the definition that not only is the usual definition logically included as a special case, but "infinity" is also included as a "limit," as well as any complex number.

Most of the suggestions made in the paper have been yearly presented to his students in calculus by the writer beginning with and since the year 1893.

It is with considerable diffidence that he presents his views, yet at the same time he feels moved to do so.

First we have to note for the purposes of these suggestions that the word "number" may have two meanings: First as an "expressible number." In this sense a number is said to be expressible when its modulus satisfies the statement

$$m < \text{mod (number)} < n,$$

where m and n are positive commensurable numbers. That is, if a number in this sense is present, two commensurable values, m and n, can always be found lying on either side of its modulus, and hemming it in, so to speak.

In this sense

0 is not a number,
ϵ is a number (ϵ being indefinitely small, but $\epsilon \neq 0$),
$1/\epsilon = \omega$ (ω being indefinitely great, but $\omega \neq \infty$),
∞ is not a number ($\infty = 1/0$).

But there is a second sense in which we may think of number as any symbol other than a symbol of operation which is capable of occupying a place in a combination or formula composed of like symbols. In this sense 0 and ∞ are also numbers, and

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we shall also desire to speak of them as numbers in this generalized sense in our suggestions as well as to distinguish them from all other numbers in the first sense of "expressible number."

Next we must note that the word "limit" is multiple valued. Thus we have "limit" of the roots of an equation, "limit" of integration, or of summation, as well as, "limit" of a variable. It is in this last sense however that we wish to consider the word. And we note here by considering the distinction of "expressible" number in the first sense, that in the "theory of limits" it is in fact already used in two senses, viz:

1. *As limit or negation of all number* that may be expressed as defined in the first sense. *This limit or negation of number is not and can not be a number in the first sense.* As such we already recognize, and use a symbol (0), for zero.

Is there not another limit or negation of expressible number? viz., 1/0, or that for which the word "infinity," and the symbol ∞, might be reserved? Thus may it not be said: 0 fails of expression because there is nothing to express. ∞ fails of expression because it is beyond expression. 0 is absolute nothing; is no magnitude. It is the limit of all indefinitely small magnitudes that may be expressed. ∞ is not nothing. It is magnitude which cannot be expressed; it is all magnitude that can exist or be thought to exist, and as such is the limit of all indefinitely great magnitudes which may be expressed. Thus 0 and ∞ are both limits of number in the first sense, for they are denials or negations of the expressibility of number, the one by denying magnitude, the other by affirming all magnitude and thus denying its expressibility, and both exist as limits placed there by thought. As there can be but one zero and one infinity, and even though they are not numbers in the first sense, they are both absolute constants of thought in their nature.

2. *As a fixed number, between which and a variable the absolute value of the difference may be and is made indefinitely small.* This limit is always a number in the first sense. It may be noted that mathematicians generally assign to zero the role of a fixed number, in this second sense of the use of the word "limit."

With the help of the unique geometrical representation of the variable in the complex plane, the following is suggested as a definition of limit, which shall not only include both these meanings as special cases, and which refer only to the real variable, but which shall include the complex variable as well, and thus both generalize and unify the treatment of the subject. In this definition zero and infinity are to be classed as numbers in the sense of the second meaning of number. The definition will include zero and infinity as limits.

DEFINITION.—*One number a is the limit of another variable number z when a relation exists between them such that under the law of its change no circle may be drawn around the first number inside of which the second may not pass and remain as it approaches the former, but without, however, absolutely and permanently reaching it.*

To elucidate how inclusive this definition is we note the following.

SPECIAL CASES.

1. If a represents a real finite constant number, we have the ordinary statement of limits satisfied, viz., that $a = Lx$, when

$$a - \alpha < x < a + \alpha, \quad \text{or} \quad \text{mod}\,(x - a) < \epsilon.$$

That is our definition becomes exactly logically equivalent with the ordinary definition. Thus it is not in conflict with it, but includes it as a special case. *Zero is also included under a.* Fig. 1.

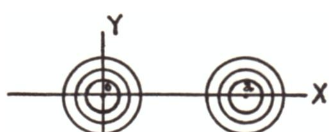

Fig. 1

2. If a represents a finite complex number, we see that our definition is logically equivalent with the following: *A finite complex number a is the limit of a variable number z when a relation exists between them such that under the law of its change the absolute value or modulus of the difference $z - a$ can be and is made to stay less than any assigned positive number ϵ however small without however becoming absolutely and permanently equal to zero.* Fig. 2.

3. If a represents ***absolute infinity*** the modulus of the difference cannot now be made indefinitely small. This then shows itself as a new case distinct from 1 and 2 but still included as a special case under the general definition. Fig. 3.

In case (3) the circles described about ∞ are parallel straight lines. The variable simply passes over them and goes off to an

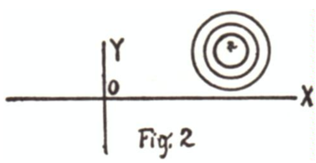

Fig. 2

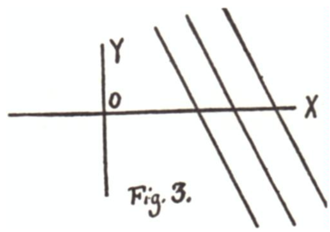

Fig. 3.

indefinitely great distance. No parallel however far away from 0 can be drawn over which the variable may not pass as it moves toward its limit ∞.

In case 1 we have $Lx = a$, including $L\epsilon = 0$.

In case 2 we have $Lz = a$.

In case 3 we have $L\omega = \infty$.

That is in case 3 we have a "limit" of the kind mentioned in (1), where we mean that ω is limited, not by any "expressible" number, it cannot be according to the definition, but only by the denial of the expressibility of number. To "a variable which increases without limit" ("limit" in the sense of (2), there is no "limit" of the kind mentioned in (2), no fixed and finite number, but there is a "limit" in the sense of 1, viz., a negation of expressibility, a negation that is required as a logical necessity of thought, an idea, or concept, just as zero is a negation, a logical necessity of thought, an idea or concept. We shall say then that 1/0 is a symbol representing magnitude, but not expressible number, indeed it is a symbol representing the negation of expressible magnitude; it is a superior limit of expressible number, a thought limit, not a numerical one, just as zero is a non-numerical inferior thought limit.

By adopting this generalized definition we have justified writing the statements:

$$\epsilon \doteq 0, \quad x \doteq a, \quad z \doteq a, \quad \omega \doteq \infty.$$

In conclusion the following observations on the definition may be made:

1. Some authors would have preferred to omit the words "but, without however, absolutely and permanently reaching it" from the definition. The writer however preferred to include them, for the sake as he considers it of not removing the props from under the foundation of a sound and logical theory of limits. On the other hand he is also glad to acknowledge that there are others who are not of his way of thinking.

A variable may pass through its limit in approaching it, but it is not permanently equal to it, as in the case of the variable $\sin x/e^x$ which passes through its limit zero infinitely many times as it approaches it, as $x \doteq \infty$.

"The limit of a constant is that constant" can always be eliminated by associating the constant with a variable by addition, subtraction, multiplication or division.

2. The definition is intended only for continuous function, or for such as whose representation by stereographic projection on the sphere would be continuous, or for the function on the side on which it is continuous (one-sided continuity). The continuous function is the only workable and usable function. It alone is subject to law and the laws of calculation. It is a loyal subject of the mathematical kingdom. Other so-called or miscalled functions are freaks, anarchists, disturbers of the peace, malformed curiosities which one and all are of no use to anyone, least of all to the loyal and burden-bearing subjects who by keeping the laws maintain the kingdom and make its advance possible. If such lawless freaks "reach their limits" or perform all sorts of acts impossible for the law abiding function, what does it signify? Least of all in order to legalize their acts shall the laws be done away with. It would seem that they should have reached their "limits" in another sense; and that scholarship lies in the direction of paying deference to the loyal continuous function rather than to the outlaws of mathematical society.

3. The definition will not be regarded with any favor by those who can not allow any vestige of a geometrical concept to run in their thoughts when they seek rigor. It is certain that the generalization cannot be made in terms of inequalities, though such will occur among the special cases. But the writer

(To be continued.)

believes it is an extreme view which would exclude every vestige of geometry from thought, because without criticism one may be misled by it. In the same way without sufficient care one may be equally misled by analysis. Error is not necessarily inherent in geometry or analysis, but in the thinking human mind.

4. By considering the special case (2), we see that $Lz = a$, may be written $L(x + yi) = \alpha + \beta i$, whence $Lx = \alpha$, $Ly = \beta$, which suggests another point of view, from which the generalization may be extended to the higher complex number. Thus if $Lx_1 = \alpha_1$, $Lx_2 = \alpha_2$, $\cdots$, $Lx_n = \alpha_n$, then $L(x_1 i_1 + x_2 i_2 + \cdots + x_n i_n) = \alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n$, in which the limit of each scalar is to be taken from the standpoint of the definition.

5. Finally the writer wishes to emphasize the fact that the suggestions made here constitute merely a point of view or one way of interpreting the facts. He has no quarrel with those who prefer to interpret them otherwise. He perfectly understands what is meant when one insists that we should write $x = \infty$, instead of $x \doteq \infty$. To him these signify exactly the same fact, only that the symbol ∞ is used with different meanings in the two statements. In the former it signifies a variable which is becoming indefinitely great. In the latter it signifies an absolute constant of thought, a negation of expressibility. In the whole presentation the writer has merely tried to explain that there is another way than that ordinarily proposed of looking at the facts involved in the theory of limits, and that instead of being in conflict with the ordinary view, it logically includes it as a special case.

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